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HEDGING CONTINGENT CLAIMS WITH CONSTRAINED PORTFOLIOS[1]

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We employ a stochastic control approach to study the question of hedging contingent claims by portfolios constrained to take values in a given closed, convex subset of $\mathscr{R}^d$. In the framework of our earlier work for utility maximization with constrained portfolios, we extend results of El Karoui and Quenez on incomplete markets and treat the case of different interest rates for borrowing and lending.

1. Introduction and summary. The celebrated papers of Black and Scholes (1973) and Merton (1973) paved the way for pricing options on stocks, based on the following principle: In a *complete market* (such as the one in Section 2 of this article) every contingent claim can be attained exactly by investing in the market and starting with a large enough initial capital. Thus, the "fair price" of the claim is taken to be the minimal such capital. This, in turn, is shown to be equal to the expectation of the discounted value of the claim, under a new, so-called risk-neutral, probability measure [Harrison and Pliska (1981); see also Harrison and Kreps (1979) and Cox and Ross (1976)]. The argument that leads to this result, and to the associated "valuation formulae", is by now standard [e.g., Karatzas and Shreve (1988) and Karatzas (1989)]; it is based on the martingale representation and Girsanov theorems from stochastic analysis, and is reviewed for the sake of completeness in Section 4 of this paper.

The foregoing argument fails, however, in an *incomplete market*, a prototypical example of which is a market in which claims can depend on stocks that are not available for investment. The option pricing problem under incompleteness of this type has been studied, among others, by Föllmer and Schweizer (1991), who adopt a risk-minimization approach, and by Ansel and Stricker (1992), Jacka (1992) and, most notably, El Karoui and Quenez (1993). Using a stochastic control approach similar to that of the latter paper, we attack here a more general problem: *the hedging of contingent claims with portfolios constrained to take values in a given closed, convex set K.* The model employed is a, by now, standard generalization of that in Merton (1969, 1973). The framework and insights of Cvitanić and Karatzas (1992), in

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which we studied the constrained portfolio optimization problem, are essential in obtaining the main results. These can be summarized as follows: Under appropriate conditions, it is possible to replicate contingent claims even with constrained portfolios, albeit some additional consumption may be necessary; the minimal initial capital that makes this replication possible is equal to the supremum of the expected discounted values of the claim under new probability measures in a suitably large family (Theorem 6.4); replication without extra consumption is possible only if the Black–Scholes replicating portfolio happens to take values in K (Theorems 6.6, 6.7); and the associated wealth process is the minimal adapted solution of a backwards stochastic differential equation with convex constraints. The main mathematical tool, namely, the martingale approach to stochastic control, is adapted from Davis and Varaiya (1973), as reported in Elliott (1982).

The paper is organized as follows: The ingredients of the model are laid out in Sections 2–5. In Section 6 we state and prove the main results. Section 7 deals with some special cases in which more explicit results can be obtained. We discuss some possible extensions, as well as the issue of numerical calculations, in Section 8. Finally, we show in Section 9 how to apply this same approach in the (unconstrained) case of a market with different interest rates for borrowing and lending. It turns out that in such a market a large class of contingent claims, including European call options, is attainable. The results of this section extend those in the recent preprint by Korn (1992); compare also with Bergman (1991) and Jouini and Kallal (1993). These results also provide a concrete example, with "explicit" solution, of an *adapted* solution to a *nonlinear*, *backwards stochastic differential equation* in the spirit of Pardoux and Peng (1990); cf. Remark 9.6 and Example 9.5. A pricing method for the problem of Section 9, using utility functions, is proposed in Barron and Jensen (1990).

2. The model. We consider a financial market $\mathscr{M}$ that consists of one *bond* and several (d) *stocks*. The prices $P_o(t)$, $\{P_i(t)\}_{1\le i\le d}$, of these financial instruments evolve according to the equations

$$dP_o(t) = P_o(t)r(t)\,dt, \qquad P_o(0) = 1, \tag{2.1}$$

$$dP_i(t) = P_i(t)\left[b_i(t)\,dt + \sum_{j=1}^{d}\sigma_{ij}(t)\,dW^{(j)}(t)\right], \tag{2.2}$$

$$P_i(0) = p_i \in (0,\infty),\ i = 1,\dots,d.$$

Here $W = (W^{(1)},\dots,W^{(d)})^*$ is a standard Brownian motion in $\mathscr{R}^d$, defined on a complete probability space $(\Omega,\mathscr{F},\mathbb{P})$, and we shall denote by $\{\mathscr{F}_t\}$ the $\mathbb{P}$-augmentation of the filtration $\mathscr{F}_t^W = \sigma(W(s);\ 0\le s\le t)$ generated by W. The *coefficients* of $\mathscr{M}$, that is, the processes $r(t)$ (scalar interest rate), $b(t) = (b_1(t),\dots,b_d(t))^*$ (vector of appreciation rates) and $\sigma(t) = \{\sigma_{ij}(t)\}_{1\le i,j\le d}$ (volatility matrix), are assumed to be progressively measurable with respect to $\{\mathscr{F}_t\}$ and *bounded* uniformly in $(t,\omega)\in[0,T]\times\Omega$. We shall also impose the following strong nondegeneracy condition on the matrix $a(t) \triangleq \sigma(t)\sigma^*(t)$:

$$\xi^* a(t)\xi \geq \varepsilon \|\xi\|^2, \quad \forall\, (t,\xi) \in [0,T] \times \mathscr{R}^d \tag{2.3}$$

almost surely, for a given real constant $\varepsilon > 0$. All processes encountered throughout the paper will be defined on the fixed, finite horizon $[0, T]$.

We introduce also the "relative risk" process

$$\theta(t) \triangleq \sigma^{-1}(t)[b(t) - r(t)\mathbf{1}], \tag{2.4}$$

where $\mathbf{1} = (1, \dots, 1)^*$. The exponential martingale

$$Z_0(t) \triangleq \exp\left[-\int_0^t \theta^*(s)\, dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2\, ds\right] \tag{2.5}$$

and the discount process

$$\gamma_0(t) \triangleq \exp\left\{-\int_0^t r(s)\, ds\right\} \tag{2.6}$$

will be employed quite frequently.

2.1 REMARK. It is a straightforward consequence of the strong nondegeneracy condition (2.3), that the matrices $\sigma(t), \sigma^*(t)$ are invertible, and that the norms of $(\sigma(t))^{-1}, (\sigma^*(t))^{-1}$ are bounded above and below by δ and $1/\delta$, respectively, for some $\delta \in (1, \infty)$; compare with Karatzas and Shreve [(1988), page 372]. The boundedness of $b(\cdot)$, $r(\cdot)$ and $(\sigma(\cdot))^{-1}$ implies that of $\theta(\cdot)$, and thus also the martingale property of the process $Z_0(\cdot)$ in (2.5).

3. Portfolio, consumption and wealth processes. Consider now an economic agent whose actions cannot affect market prices and who can decide, at any time $t \in [0, T]$, the following points:

1. What proportion $\pi_i(t)$ of his wealth $X(t)$ to invest in the ith stock $(1 \leq i \leq d)$.
2. What amount of money $c(t+h) - c(t) \geq 0$ to withdraw for consumption during the interval $(t, t+h]$, $h > 0$.

Of course these decisions can only be based on the current information $\mathscr{F}_t$, without anticipation of the future. With $\pi(t) = (\pi_1(t), \dots, \pi_d(t))^*$ chosen, the amount $X(t)[1 - \sum_{i=1}^d \pi_i(t)]$ is invested in the bond. Thus, in accordance with the model set forth in (2.1) and (2.2), the wealth process $X(t)$ satisfies the linear stochastic equation

$$\begin{aligned} dX(t) &= \sum_{i=1}^d \pi_i(t) X(t) \left\{ b_i(t)\, dt + \sum_{j=1}^d \sigma_{ij}(t)\, dW^{(j)}(t) \right\} \\ &\quad + \left\{ 1 - \sum_{i=1}^d \pi_i(t) \right\} X(t) r(t)\, dt - dc(t) \\ &= r(t) X(t)\, dt - dc(t) \\ &\quad + X(t)\pi^*(t)\sigma(t)\, dW_0(t), \quad X(0) = x > 0, \end{aligned} \tag{3.1}$$

where the real number $x > 0$ represents initial capital, $c(t)$ is the cumulative consumption up to time t and

$$W_0(t) \triangleq W(t) + \int_0^t \theta(s)\, ds, \qquad 0 \le t \le T. \tag{3.2}$$

We formalize the preceding discussion as follows.

3.1 DEFINITION. (i) An $\mathscr{R}^d$-valued, $\{\mathscr{F}_t\}$-progressively measurable process $\pi = \{\pi(t),\ 0 \le t \le T\}$ with $\int_0^T \|\pi(t)\|^2\, dt < \infty$ a.s. will be called a *portfolio process*.

(ii) A nonnegative, nondecreasing, $\{\mathscr{F}_t\}$-progressively measurable process $c = \{c(t),\ 0 \le t \le T\}$ with RCLL paths, $c(0) = 0$ and $c(T) < \infty$ a.s. will be called a *consumption process*.

(iii) Given a pair (π, c) as before, the solution $X \equiv X^{x,\pi,c}$ of (3.1) will be called the *wealth process* corresponding to the portfolio–consumption pair (π,c) and initial capital $x \in (0,\infty)$.

3.2 DEFINITION. A portfolio–consumption process pair (π, c) is called *admissible* for the initial capital $x \in (0,\infty)$, if

$$X^{x,\pi,c}(t) \ge 0, \qquad \forall\, 0 \le t \le T, \tag{3.3}$$

holds almost surely. The set of admissible pairs (π, c) will be denoted by $\mathscr{A}_0(x)$.

In the notation (2.5) and (2.6), (3.1) leads to

$$\begin{aligned} M_0(t) &\triangleq \gamma_0(t)X(t) + \int_0^t \gamma_0(s)\, dc(s) \\ &= x + \int_0^t \gamma_0(s)X(s)\pi^*(s)\sigma(s)\, dW_0(s). \end{aligned} \tag{3.4}$$

In particular, the process $M_0(\cdot)$ of (3.4) is seen to be a continuous local martingale under the so-called risk-neutral probability measure (or "equivalent martingale measure")

$$\mathbb{P}^0(A) \triangleq E[Z_0(T)1_A], \qquad A \in \mathscr{F}_T. \tag{3.5}$$

If $(\pi, c) \in \mathscr{A}_0(x)$, the $\mathbb{P}^0$-local martingale $M_0(\cdot)$ of (3.4) is also nonnegative, thus a supermartingale. Consequently,

$$E^0\left[\gamma_0(T)X^{x,\pi,c}(T) + \int_0^T \gamma_0(t)\, dc(t)\right] \le x, \quad \forall\, (\pi, c) \in \mathscr{A}_0(x). \tag{3.6}$$

Here, E^0 denotes the expectation operator under the measure $\mathbb{P}^0$. Under this measure, the process W_0 of (3.2) is standard Brownian motion by the Girsanov theorem [e.g., Karatzas and Shreve (1988), Section 3.5] and the discounted stock prices $\gamma_0(\cdot)P_i(\cdot)$ are martingales, because

$$dP_i(t) = P_i(t)\left[r(t)\,dt + \sum_{j=1}^{d} \sigma_{ij}(t)\,dW_0^{(j)}(t)\right], \tag{3.7}$$
$$P_i(0) = p_i,\ i = 1,\dots,d,$$

from (2.2) and (3.2).

3.3 Remark. For any given $(\pi, c) \in \mathscr{A}_0(x)$, let $X(\cdot) \equiv X^{x,\pi,c}(\cdot)$ and define the "bankruptcy time"

$$S \triangleq \inf\{t \in [0,T];\ X(t) = 0\} \wedge T. \tag{3.8}$$

Because the continuous process $M_0(\cdot)$ of (3.4) is a $\mathbb{P}^0$-supermartingale, the same is true of $\gamma_0(\cdot)X(\cdot)$ and thus (e.g. Karatzas and Shreve (1988), Problem 1.3.29) for a.e. $\omega \in \{S < T\}$,

$$X(t, \omega) = 0, \qquad \forall\, t \in [S(\omega), T]. \tag{3.9}$$

On the other hand, the optional sampling theorem applied to $M_0(\cdot)$ yields

$$E^0\left[\gamma_0(T)X(T) + \int_S^T \gamma_0(t)\,dc(t)\middle|\mathscr{F}_S\right] \le \gamma_0(S)X(S) \quad \text{a.s.}$$

and thus for a.e. $\omega \in \{S < T\}$,

$$c(t, \omega) = c(S(\omega), \omega), \qquad \ell-\text{a.e. } t \in (S(\omega), T], \tag{3.10}$$

where ℓ denotes "Lebesgue measure". It follows from (3.9) and (3.10) that bankruptcy is an absorbing state for $(\pi, c) \in \mathscr{A}_0(x)$. If the wealth $X(\cdot)$ becomes equal to zero before the end T of the horizon, it stays there; no further consumption takes place, and further values of the portfolio $\pi(\cdot)$ become irrelevant. In fact, we will allow the possibility of bankruptcy even if there is no consumption at all; that is, we will allow wealth processes modeled by (3.1) [possibly with $c(\cdot) \equiv 0$] for $t < S$, where S is some stopping time and $X(\cdot) = 0$ for $S \le t \le T$.

4. Hedging with unconstrained portfolios. Let us suppose now that an agent promises to pay a random amount $B(\omega) \ge 0$ at time $t = T$. *What is the value of this promise at time* $t = 0$? In other words, how much should the agent charge for selling a contractual obligation that entitles its holder to a payment of size $B(\omega)$ at $t = T$?

For instance, suppose that this obligation stipulates selling one share of the first stock at a contractually specified price q. If at time $t = T$ the price $P_1(T, \omega)$ of the stock is below q, the contract is worthless to its holder; if not, the holder can purchase the stock at the price q per share and then sell it at price $P_1(T, \omega)$, thus making a profit of $P_1(T, \omega) - q$. In other words, this contract entitles its holder to a payment of $B(\omega) = (P_1(T, \omega) - q)^+$ at time $t = T$; it is called a (European) *call option* with "exercise price" q and "maturity date" T.

To answer the question of the first paragraph, one argues as follows. Suppose the agent sets aside an amount $x > 0$ at time $t = 0$; he invests in the market $\mathscr{M}$ according to some portfolio $\pi(\cdot)$ and withdraws (possibly) funds according to a cumulative consumption process $c(\cdot)$, but wants to be certain that at time $t = T$ he will be able to *cover his obligation*, that is, that $X^{x,\pi,c}(T) \geq B$ will hold almost surely. What is the smallest value of $x > 0$ for which such "hedging" is possible? This smallest value will then be the "price" of the contract at time $t = 0$.

4.1 DEFINITION. A *contingent claim* is a nonnegative, $\mathscr{F}_T$-measurable random variable B that satisfies

$$0 < E^0[\gamma_0(T)B] < \infty. \tag{4.1}$$

The *hedging price* of this contingent claim is defined by

$$u_0 \triangleq \inf\{x > 0; \exists\, (\pi, c) \in \mathscr{A}_0(x) \text{ s.t. } X^{x,\pi,c}(T) \geq B \text{ a.s.}\}. \tag{4.2}$$

The following "classical" result identifies u_0 as the expectation, under the risk-neutral probability measure of (3.5), of the claim's discounted value.

4.2 PROPOSITION. *The infimum in* (4.2) *is attained, and we have*

$$u_0 = E^0[\gamma_0(T)B]. \tag{4.3}$$

Furthermore, there exists a portfolio $\pi_0(\cdot)$ *such that* $X_0(\cdot) \equiv X^{u_0,\pi_0,0}(\cdot)$ *is given by*

$$X_0(t) = \frac{1}{\gamma_0(t)} E^0[\gamma_0(T)B \mid \mathscr{F}_t], \qquad 0 \leq t \leq T. \tag{4.4}$$

PROOF. Suppose $X^{x,\pi,c}(T) \geq B$ holds a.s. for some $x \in (0, \infty)$ and a suitable pair $(\pi, c) \in \mathscr{A}_0(x)$. Then from (3.6) we have $x \geq z \triangleq E^0[\gamma_0(T)B]$ and thus $u_0 \geq z$.

On the other hand, from the martingale representation theorem and one of its well-known variants [cf. Karatzas and Shreve (1988), pages 182–184 and 375], the process

$$X_0(t) \triangleq \frac{1}{\gamma_0(t)} E^0[\gamma_0(T)B \mid \mathscr{F}_t], \qquad 0 \leq t \leq T,$$

can be represented as

$$X_0(t) = \frac{1}{\gamma_0(t)} \left[z + \int_0^t \psi^*(s)\, dW_0(s) \right] \tag{4.5}$$

for a suitable $\{\mathcal{F}_t\}$-progressively measurable process $\psi(\cdot)$ with values in $\mathscr{R}^d$ and $\int_0^T \|\psi(t)\|^2\,dt < \infty$ a.s. Then

$$\pi_0(t) \triangleq \frac{1}{\gamma_0(t)X_0(t)}(\sigma^*(t))^{-1}\psi(t)$$

is a well-defined portfolio process (recall Remarks 2.1 and 3.3) and a comparison of (4.5) with (3.4) yields $X_0(\cdot) \equiv X^{z,\pi_0,0}(\cdot)$. Therefore, $z \geq u_0$. □

In the sequel, we shall refer to u_0, $X_0(\cdot)$ and $\pi_0(\cdot)$ as the unconstrained hedging price, price process, and portfolio, respectively. It should be noticed that

$$X_0(T) = X_0^{u_0,\pi_0,0}(T) = B \quad \text{a.s.} \tag{4.6}$$

in Theorem 4.2. We express this by saying that the contingent claim is *attainable* (with initial capital u_0, portfolio π_0 and zero consumption).

4.3 EXAMPLE. *Constant $r(\cdot) \equiv r > 0$ and $\sigma(\cdot) \equiv \sigma$ nonsingular.* In this case, the solution $P(t) = (P_1(t),\dots,P_d(t))^*$ is given by $P_i(t) = h_i(t-s, P(s), \sigma(W_0(t) - W_0(s)))$, $0 \leq s \leq t$, where $h\colon [0,\infty) \times \mathscr{R}_+^d \times \mathscr{R}^d \to \mathscr{R}_+^d$ is the function defined by

$$h_i(t,p,y;r) \triangleq p_i \exp\left[\left(r - \tfrac{1}{2}a_{ii}\right)t + y_i\right], \qquad i = 1,\dots,d. \tag{4.7}$$

Consider now a contingent claim of the type $B = \varphi(P(T))$, where $\varphi\colon \mathscr{R}_+^d \to [0,\infty)$ is a given continuous function that satisfies polynomial growth conditions in both $\|p\|$ and $1/\|p\|$. Then it is rather straightforward, using the Feynman–Kac theorem [e.g., Karatzas and Shreve (1988), page 366] and Itô's rule, to see that the processes $X_0(\cdot)$ and $\pi_0(\cdot)$ of Proposition 4.4 are given as

$$X_0(t) = e^{-r(T-t)}E^0[\varphi(P(T))|\mathcal{F}_t] = U(T-t, P(t)), \tag{4.8}$$

$$\pi_{0i}(t) = \frac{P_i(t)(\partial/\partial p_i)U(T-t,P(t))}{U(T-t,P(t))}, \qquad i = 1,\dots,d, \tag{4.9}$$

respectively, where

$$U(t,p) \triangleq \begin{cases} e^{-rt}\int_{\mathscr{R}^d} \varphi(h(t,p,\sigma z;r))\dfrac{e^{-\|z\|^2/2t}}{(2\pi t)^{d/2}}\,dz, & t > 0,\ p \in \mathscr{R}_+^d,\\ \varphi(p), & t = 0,\ p \in \mathscr{R}_+^d. \end{cases} \tag{4.10}$$

In particular, the unconstrained hedging price u_0 of (4.3) is given, in terms of the function U of (4.10), by

$$u_0 = X_0(0) = U(T, P(0)). \tag{4.11}$$

A very explicit computation for the function U is possible for $d = 1$ in the case $\varphi(p) = (p-q)^+$ of a *call option*: with $\sigma = \sigma_{11} > 0$, exercise price $q > 0$, $\Phi(z) = (1/\sqrt{2\pi})\int_{-\infty}^z e^{-u^2/2}\,du$ and

$$\nu_{\pm}(t,p) \triangleq (1/\sigma\sqrt{t})\left[\log(p/q) + (r \pm \sigma^2/2)t\right],$$

we have the famous *Black and Scholes* (1973) *formula*:

$$U(t,p) = \begin{cases} p\Phi(\nu_+(t,p)) - qe^{-rt}\Phi(\nu_-(t,p)), & t > 0,\ p \in (0,\infty), \\ (p-q)^+, & t = 0,\ p \in (0,\infty). \end{cases} \tag{4.12}$$

5. Convex sets and constrained portfolios. We shall fix throughout a nonempty, closed, convex set K in $\mathscr{R}^d$, and denote by

$$\delta(x) \equiv \delta(x|K) \triangleq \sup_{\pi \in K} (-\pi^* x) : \mathscr{R}^d \to \mathscr{R} \cup \{+\infty\} \tag{5.1}$$

the support function of the convex set $-K$. This is a closed, positively homogeneous, proper convex function on $\mathscr{R}^d$ [Rockafellar (1970), page 114]. It is finite on its *effective domain*

$$\begin{aligned} \tilde{K} &\triangleq \{x \in \mathscr{R}^d;\ \delta(x|K) < \infty\} \\ &= \{x \in \mathscr{R}^d;\ \exists\, \beta \in \mathscr{R} \text{ s.t. } -\pi^* x \le \beta,\ \forall\, \pi \in K\}, \end{aligned} \tag{5.2}$$

which is a convex cone (called the barrier cone of $-K$). It will be assumed throughout this paper that the function

$$\delta(\cdot|K) \text{ is continuous on } \tilde{K} \tag{5.3}$$

and bounded from below on $\mathscr{R}^d$:

$$\delta(x|K) \ge \delta_0, \qquad \forall\, x \in \mathscr{R}^d \quad \text{for some } \delta_0 \in \mathscr{R}. \tag{5.4}$$

5.1 REMARK. Condition (5.4) is obviously satisfied (with $\delta_0 = 0$) if K contains the origin. On the other hand, Theorem 10.2 in Rockafellar [(1970), page 84] guarantees that (5.3) is satisfied, in particular, if $\tilde{K}$ is locally simplicial.

5.2 EXAMPLES. The role of the closed, convex set K that we just introduced is to model reasonable constraints on portfolio choice. One may, for instance, consider the following examples, all of which satisfy the conditions (5.3) and (5.4):

(i) *Unconstrained case*: $K = \mathscr{R}^d$. Then $\tilde{K} = \{0\}$ and $\delta \equiv 0$ on $\tilde{K}$.

(ii) *Prohibition of short-selling*: $K = [0,\infty)^d$. Then $\tilde{K} = K$ and $\delta \equiv 0$ on $\tilde{K}$.

(iii) *Incomplete market*: $K = \{\pi \in \mathscr{R}^d;\ \pi_i = 0,\ \forall\ i = m+1,\dots,d\}$ for some fixed $m \in \{1,\dots,d-1\}$. Then $\tilde{K} = \{x \in \mathscr{R}^d;\ x_i = 0,\ \forall\ i = 1,\dots,m\}$ and $\delta \equiv 0$ on $\tilde{K}$.

(iv) *Incomplete market with prohibition of short-selling*: $K = \{\pi \in \mathscr{R}^d;\ \pi_i \ge 0,\ \forall\ i = 1,\dots,m$ and $\pi_i = 0,\ \forall\ i = m+1,\dots,d\}$ with m as in (iii). Then $\tilde{K} = \{x \in \mathscr{R}^d;\ x_i \ge 0,\ \forall\ i = 1,\dots,m\}$ and $\delta \equiv 0$ on $\tilde{K}$.

(v) *K is a closed, convex cone in $\mathscr{R}^d$*. Then $\tilde{K} = \{x \in \mathscr{R}^d;\ \pi^* x \ge 0,\ \forall\ \pi \in K\}$ is the polar cone of $-K$ and $\delta \equiv 0$ on $\tilde{K}$. This case obviously generalizes (i)–(iv).

(vi) *Prohibition of borrowing*: $K = \{\pi \in \mathscr{R}^d;\ \sum_{i=1}^d \pi_i \le 1\}$. Then $\tilde{K} = \{x \in \mathscr{R}^d;\ x_1 = \cdots = x_d \le 0\}$ and $\delta(x) = -x_1$ on $\tilde{K}$.

(vii) *Rectangular constraints*: $K = \times_{i=1}^{d} I_i$, $I_i = [\alpha_i, \beta_i]$ for some fixed numbers $-\infty \le \alpha_i \le 0 \le \beta_i \le \infty$, with the understanding that the interval I_i is open to the right (left) if $b_i = \infty$ (respectively, if $\alpha_i = -\infty$). Then $\delta(x) = \sum_{i=1}^{d} (\beta_i x_i^- - \alpha_i x_i^+)$ and $\tilde{K} = \mathscr{R}^d$ if all the α_i's and β_i's are real. In general, $\tilde{K} = \{x \in \mathscr{R}^d;\ x_i \ge 0,\ \forall\ i \in \mathscr{S}_+ \text{ and } x_j \le 0,\ \forall\ j \in \mathscr{S}_-\}$, where $\mathscr{S}_+ \triangleq \{i = 1, \dots, d/\beta_i = \infty\}$ and $\mathscr{S}_- \triangleq \{i = 1, \dots, d/\alpha_i = -\infty\}$.

From now on, we consider only portfolios that take values in the given, convex, closed set $K \subset \mathscr{R}^d$; that is, we replace the set of admissible policies $\mathscr{A}_0(x)$ with

$$(5.5)\qquad \mathscr{A}'(x) \triangleq \{(\pi, c) \in \mathscr{A}_0(x);\ \pi(t,\omega) \in K \text{ for } \ell \otimes \mathbb{P}\text{-a.e. } (t,\omega)\}.$$

As in Cvitanić and Karatzas (1992), hereafter abbreviated as CK, consider the class $\mathscr{H}$ of $\tilde{K}$-valued, $\{\mathscr{F}_t\}$-progressively measurable processes $\nu = \{\nu(t), 0 \le t \le T\}$ that satisfy $E\int_0^T \|\nu(t)\|^2\,dt + E\int_0^T \delta(\nu(t))\,dt < \infty$, and introduce for every $\nu \in \mathscr{H}$ the analogues

$$(5.6)\qquad \theta_\nu(t) \triangleq \theta(t) + \sigma^{-1}(t)\nu(t),$$

$$(5.7)\qquad \gamma_\nu(t) \triangleq \exp\left[-\int_0^t \{r(s) + \delta(\nu(s))\}\,ds\right],$$

$$(5.8)\qquad Z_\nu(t) \triangleq \exp\left[-\int_0^t \theta_\nu^*(s)\,dW(s) - \frac{1}{2}\int_0^t \|\theta_\nu(s)\|^2\,ds\right],$$

$$(5.9)\qquad W_\nu(t) \triangleq W(t) + \int_0^t \theta_\nu(s)\,ds$$

of the processes in (2.4)–(2.6) and (3.2), as well as the measure

$$(5.10)\qquad \mathbb{P}^\nu(A) \triangleq E[Z_\nu(T)1_A] = E^\nu[1_A], \qquad A \in \mathscr{F}_T,$$

by analogy with (3.5). Finally, denote by $\mathscr{D}$ the subset consisting of the processes $\nu \in \mathscr{H}$ for which the exponential local martingale $Z_\nu(\cdot)$ of (5.8) is actually a martingale. Thus, for every $\nu \in \mathscr{D}$, the measure $\mathbb{P}^\nu$ of (5.10) is a probability measure and the process $W_\nu(\cdot)$ of (5.9) is a $\mathbb{P}^\nu$-Brownian motion.

5.3 Definition. A contingent claim B will be called *K-hedgeable* if it satisfies

$$(5.11)\qquad V(0) \triangleq \sup_{\nu \in \mathscr{D}} E^\nu[\gamma_\nu(T)B] < \infty.$$

This definition will be justified in the next section. More precisely, it will be shown there that for any K-hedgeable contingent claim B, there exists a pair $(\pi, c) \in \mathscr{A}'(V(0))$ such that $X^{V(0),\pi,c}(T) = B$, and that $V(0)$ is the minimal initial wealth for which this can be achieved.

5.4 Remark. In the unconstrained case $K = \mathscr{R}^d$ we have $\tilde{K} = \{0\}$, and $V(0) = E^0[\gamma_0(T)B]$ is then the unconstrained hedging price for the contingent claim B, as in Proposition 4.2. In the framework of CK, the number $u_\nu \triangleq E^\nu[\gamma_\nu(T)B] = E[\gamma_\nu(T)Z_\nu(T)B]$ is the unconstrained hedging price for B in an *auxiliary market* $\mathscr{M}_\nu$. This market consists of a bond with interest rate $r^{(\nu)}(t) \triangleq r(t) + \delta(\nu(t))$ and d stocks, with the same volatility matrix $\{\sigma_{ij}(t)\}_{1 \le i,\ j \le d}$ as before and appreciation rates $b_i^{(\nu)}(t) \triangleq b_i(t) + \nu_i(t) + \delta(\nu(t))$, $1 \le i \le d$, for any given $\nu \in \mathscr{D}$. Thus, the prices of these instruments in $\mathscr{M}_\nu$ are given, by analogy with (2.1) and (2.2), as

$$dP_o^{(\nu)}(t) = P_o^{(\nu)}(t)[r(t) + \delta(\nu(t))]\,dt, \qquad P_o^{(\nu)}(0) = 1, \tag{5.12}$$

$$\begin{aligned} dP_i^{(\nu)}(t) &= P_i^{(\nu)}(t)\Bigg[\{b_i(t) + \nu_i(t) + \delta(\nu(t))\}\,dt \\ &\qquad + \sum_{j=1}^{d} \sigma_{ij}(t)\,dW^{(j)}(t)\Bigg], \\ P_i^{(\nu)}(0) &= p_i \in (0,\infty),\ i = 1,\dots,d. \end{aligned} \tag{5.13}$$

We shall show that the price for hedging B with a constrained portfolio in the market $\mathscr{M}$ is given by the supremum of the unconstrained hedging prices $u_\nu = E^\nu[\gamma_\nu(T)B]$ in these auxiliary markets $\mathscr{M}_\nu$, $\nu \in \mathscr{D}$.

5.5 Remark. From Theorem 9.1 in CK we know that if the supremum in (5.11) is achieved by some $\lambda \in \mathscr{D}$, then the claim B *can be hedged* by a constrained portfolio π and *without consumption* [i.e., with $c(\cdot) \equiv 0$]. This, however, will *not* always be the case: The supremum in (5.11) is not attained in general and neither is the supremum of $u_\nu = E[\gamma_\nu(T)Z_\nu(T)B]$ over the class $\mathscr{H}$, which is larger than $\mathscr{D}$. The situation should be contrasted with the portfolio optimization problem of CK, where such an enlargement *does* produce an existence result.

Let us mention, however, that the suprema of $u_\nu = E[\gamma_\nu(T)Z_\nu(T)B]$ over the two classes $\mathscr{D}$ and $\mathscr{H}$ are the same; compare with Remark 6.11.

5.6 Remark. In terms of the $\mathbb{P}^\nu$-Brownian motion $W_\nu(\cdot)$ of (5.9), the stock price equations (2.2) can be rewritten as

$$\begin{aligned} dP_i(t) &= P_i(t)\Bigg[(r(t) - \nu_i(t))\,dt \\ &\qquad + \sum_{j=1}^{d} \sigma_{ij}(t)\,dW_\nu^{(j)}(t)\Bigg], \qquad i = 1,\dots,d, \end{aligned} \tag{5.14}$$

for any given $\nu \in \mathscr{D}$.

6. Hedging with constrained portfolios. We introduce in this section the "hedging price" of a contingent claim B, with portfolios constrained to take values in the set K of Sections 4 and 5, and show that it coincides with the number $V(0) = \sup_{\nu \in \mathscr{D}} E^{\nu}[\gamma_\nu(T)B]$ of (5.11).

6.1 DEFINITION. The *hedging price with K-constrained portfolios* of a contingent claim B is defined by

$$h(0) \triangleq \begin{cases} \inf\{x \in (0,\infty); \exists\,(\pi, c) \in \mathscr{A}'(x), \text{ s.t. } X^{x,\pi,c}(T) \geq B \text{ a.s.}\}, \\ \infty, \quad \text{if the above set is empty.} \end{cases} \tag{6.1}$$

Let us denote by $\mathscr{S}$ the set of all $\{\mathscr{F}_t\}$-stopping times τ with values on $[0, T]$ and by $\mathscr{S}_{\rho,\sigma}$ the subset of $\mathscr{S}$ consisting of stopping times τ s.t. $\rho(\omega) \leq \tau(\omega) \leq \sigma(\omega)$, $\forall\ \omega \in \Omega$, for any two $\rho \in \mathscr{S}$, $\sigma \in \mathscr{S}$ such that $\rho \leq \sigma$ a.s. For every $\tau \in \mathscr{S}$ consider also the $\mathscr{F}_\tau$-measurable random variable

$$V(\tau) \triangleq \operatorname*{ess\,sup}_{\nu \in \mathscr{D}} E^{\nu}\left[B\gamma_0(T)\exp\left\{-\int_\tau^T \delta(\nu(s))\,ds\right\}\middle|\,\mathscr{F}_\tau\right] \tag{6.2}$$

[notice the notational agreement with the definition (5.11)].

6.2 PROPOSITION. *For any contingent claim that satisfies* (5.11), *the family* (6.2) *of random variables* $\{V(\tau)\}_{\tau \in \mathscr{S}}$ *satisfies the equation of dynamic programming*:

$$V(\tau) = \operatorname*{ess\,sup}_{\nu \in \mathscr{D}_{\tau,\theta}} E^{\nu}\left[V(\theta)\exp\left\{-\int_\tau^\theta \delta(\nu(u))\,du\right\}\middle|\,\mathscr{F}_\tau\right], \quad \forall\,\theta \in \mathscr{S}_{\tau,T}, \tag{6.3}$$

where $\mathscr{D}_{\tau,\theta}$ *is the restriction of* $\mathscr{D}$ *to the stochastic interval* $[\![\tau, \theta]\!]$.

6.3 PROPOSITION. *The process* $V = \{V(t), \mathscr{F}_t;\ 0 \leq t \leq T\}$ *of Proposition* 6.2 *can be considered in its RCLL modification, and for every* $\nu \in \mathscr{D}$,

$$\left\{\begin{matrix} Q_\nu(t) \triangleq V(t)\exp\left(-\int_0^t \delta(\nu(u))\,du\right), \mathscr{F}_t; 0 \leq t \leq T, \\ \text{is a } \mathbb{P}^{\nu}\text{-supermartingale with RCLL paths} \end{matrix}\right\}. \tag{6.4}$$

Furthermore, V is the smallest adapted RCLL process that satisfies (6.4) *as well as*

$$V(T) = B\gamma_0(T) \quad a.s. \tag{6.5}$$

The proofs of Propositions 6.2 and 6.3 will be given in the Appendix. The following theorem can be regarded as the main result of the paper; it justifies Definition 5.3.

6.4 THEOREM. *For an arbitrary contingent claim* B, *we have* $h(0) = V(0)$. *Furthermore, if* $V(0) < \infty$, *there exists a pair* $(\hat{\pi}, \hat{c}) \in \mathscr{A}'(V(0))$ *such that* $X^{V(0),\hat{\pi},\hat{c}}(T) = B$, *a.s.*

PROOF. We first want to show $h(0) \le V(0)$. Clearly, we may assume $V(0) < \infty$. From (6.4), the martingale representation theorem (cf. Proof of Proposition 4.2) and the Doob–Meyer decomposition [e.g., Karatzas and Shreve (1988), Section 1.4], we have for every $\nu \in \mathscr{D}$:

$$Q_\nu(t) = V(0) + \int_0^t \psi_\nu^*(s)\, dW_\nu(s) - A_\nu(t), \qquad 0 \le t \le T, \tag{6.6}$$

where $\psi_\nu(\cdot)$ is an $\mathscr{R}^d$-valued, $\{\mathscr{F}_t\}$-progressively measurable and a.s. square-integrable process and $A_\nu(\cdot)$ is adapted with increasing RCLL paths and $A_\nu(0) = 0$, $A_\nu(T) < \infty$ a.s. The idea then is to consider the positive, adapted RCLL process

$$\hat{X}(t) \triangleq \frac{V(t)}{\gamma_0(t)} = \frac{Q_\nu(t)}{\gamma_\nu(t)}, \qquad 0 \le t \le T, \qquad \forall\, \nu \in \mathscr{D}, \tag{6.7}$$

with $\hat{X}(0) = V(0)$, $\hat{X}(T) = B$ a.s. and to find a pair $(\hat{\pi}, \hat{c}) \in \mathscr{A}'(V(0))$ such that $\hat{X}(\cdot) = X^{V(0),\hat{\pi},\hat{c}}(\cdot)$. This will prove that $h(0) \le V(0)$.

In order to do this, let us observe that for any $\mu \in \mathscr{D}$, $\nu \in \mathscr{D}$, we have from (6.4),

$$Q_\mu(t) = Q_\nu(t) \exp\left[\int_0^t \{\delta(\nu(s)) - \delta(\mu(s))\}\, ds\right]$$

and from (6.6),

$$\begin{aligned} dQ_\mu(t) &= \exp\left[\int_0^t \{\delta(\nu(s)) - \delta(\mu(s))\}\, ds\right] \\ &\quad \times \left[Q_\nu(t)\{\delta(\nu(t)) - \delta(\mu(t))\}\, dt + \psi_\nu^*(t)\, dW_\nu(t) - dA_\nu(t)\right] \\ &= \exp\left[\int_0^t \{\delta(\nu(s)) - \delta(\mu(s))\}\, ds\right] \\ &\quad \times \left[\hat{X}(t)\gamma_\nu(t)\{\delta(\nu(t)) - \delta(\mu(t))\}\, dt - dA_\nu(t)\right. \\ &\qquad \left. + \psi_\nu^*(t)\sigma^{-1}(t)(\nu(t) - \mu(t))\, dt + \psi_\nu^*(t)\, dW_\mu(t)\right]. \end{aligned} \tag{6.8}$$

Comparing this decomposition with

$$dQ_\mu(t) = \psi_\mu^*(t)\, dW_\mu(t) - dA_\mu(t), \tag{6.6$'$}$$

we conclude that

$$\psi_\nu^*(t) \exp\left(\int_0^t \delta(\nu(s))\, ds\right) = \psi_\mu^*(t) \exp\left(\int_0^t \delta(\mu(s))\, ds\right)$$

and hence that this expression is independent of $\nu \in \mathscr{D}$:

$$\begin{aligned} &\psi_\nu^*(t) \exp\left(\int_0^t \delta(\nu(s))\, ds\right) \\ &\qquad = \hat{X}(t)\gamma_0(t)\hat{\pi}^*(t)\sigma(t), \quad \forall\, 0 \le t \le T,\ \nu \in \mathscr{D}, \end{aligned} \tag{6.9}$$

for some adapted, $\mathscr{R}^d$-valued, a.s. square-integrable process $\hat{\pi}$ (we do not know yet that $\hat{\pi}$ takes values in K).

Similarly, we conclude from (6.8), (6.9) and (6.6′) that

$$\exp\left(\int_0^t \delta(\nu(s))\,ds\right) dA_\nu(t) - \gamma_0(t)\hat{X}(t)[\delta(\nu(t)) + \hat{\pi}^*(t)\nu(t)]\,dt$$
$$= \exp\left(\int_0^t \delta(\mu(s))\,ds\right) dA_\mu(t) - \gamma_0(t)\hat{X}(t)[\delta(\mu(t)) + \hat{\pi}^*(t)\mu(t)]\,dt,$$

and hence this expression is also independent of $\nu \in \mathscr{D}$:

$$(6.10)\quad \hat{c}(t) \triangleq \int_0^t \gamma_\nu^{-1}(s)\,dA_\nu(s) - \int_0^t \hat{X}(s)[\delta(\nu(s)) + \nu^*(s)\hat{\pi}(s)]\,ds$$

for every $0 \le t \le T$, $\nu \in \mathscr{D}$. From (6.10) with $\nu \equiv 0$ we obtain $\hat{c}(t) = \int_0^t \gamma_0^{-1}(s)\,dA_0(s)$, $0 \le t \le T$, and hence

$$(6.11)\qquad \left\{\begin{array}{c} \hat{c}(\cdot) \text{ is an increasing, adapted, RCLL process} \\ \text{with } \hat{c}(0) = 0 \text{ and } \hat{c}(T) < \infty \text{ a.s.} \end{array}\right\}.$$

Next, we claim that

$$(6.12)\qquad \delta(\nu(t,\omega)) + \nu^*(t,\omega)\hat{\pi}(t,\omega) \ge 0, \qquad \ell \otimes \mathbb{P}\text{-a.e.},$$

holds for every $\nu \in \mathscr{D}$. Then the arguments of CK, Theorem 9.1, lead to the fact that

$$(6.12')\qquad \hat{\pi}(t,\omega) \in K \text{ holds } \ell \otimes \mathbb{P}\text{-a.e. on } [0,T] \times \Omega.$$

[These arguments need the continuity condition (5.3) and the assumption that the set K is closed.] In order to verify (6.12), notice that from (6.10) we obtain

$$A_\nu(t) = \int_0^t \gamma_\nu(s)\{d\hat{c}(s) + \hat{X}(s)\{\delta(\nu(s)) + \nu^*(s)\hat{\pi}(s)\}\,ds\}$$
$$\le k\left[\hat{c}(t) + \int_0^t \{\delta(\nu(s)) + \nu^*(s)\hat{\pi}(s)\}\hat{X}(s)\,ds\right],\ 0 \le t \le T,\ \nu \in \mathscr{D},$$

for some $k > 0$. Fix $\nu \in \mathscr{D}$ and define the set $F_t \triangleq \{\omega \in \Omega;\ \delta(\nu(t,\omega)) + \nu^*(t,\omega)\hat{\pi}(t,\omega) < 0\}$ for every $t \in [0,T]$. Let $\mu(t) \triangleq [\nu(t)1_{F_t^c} + n\nu(t)1_{F_t}](1 + \|\nu(t)\|)^{-1}$, $n \in \mathbb{N}$. Then $\mu \in \mathscr{D}$ and, assuming that (6.12) does not hold, we get for n large enough,

$$E[A_\mu(T)] \le E\left[k\hat{c}(T) + k\int_0^T (1 + \|\nu(t)\|)^{-1}\hat{X}(t)1_{F_t^c}\right.$$
$$\left.\times\{\delta(\nu(t)) + \nu^*(t)\hat{\pi}(t)\}\,dt\right]$$
$$+ nE\left[k\int_0^T (1 + \|\nu(t)\|)^{-1}\hat{X}(t)1_{F_t}\{\delta(\nu(t)) + \nu^*(t)\hat{\pi}(t)\}\,dt\right] < 0,$$

a contradiction.

Now we can put together (6.6)–(6.10) to deduce

$$
\begin{aligned}
d\big(\gamma_\nu(t)\hat{X}(t)\big) &= dQ_\nu(t) = \psi_\nu^*(t)\,dW_\nu(t) - dA_\nu(t)\\
&= \gamma_\nu(t)\big[-d\hat{c}(t) - \hat{X}(t)\{\delta(\nu(t)) + \nu^*(t)\hat{\pi}(t)\}\,dt\\
&\qquad\qquad + \hat{X}(t)\hat{\pi}^*(t)\sigma(t)\,dW_\nu(t)\big]
\end{aligned}
\tag{6.13}
$$

for any given $\nu \in \mathscr{D}$. As a consequence, the process

$$
\begin{aligned}
\hat{M}_\nu(t) &\triangleq \gamma_\nu(t)\hat{X}(t) + \int_0^t \gamma_\nu(s)\,d\hat{c}(s)\\
&\quad + \int_0^t \gamma_\nu(s)\hat{X}(s)[\delta(\nu(s)) + \nu^*(s)\hat{\pi}(s)]\,ds\\
&= V(0) + \int_0^t \gamma_\nu(s)\hat{X}(s)\hat{\pi}^*(s)\sigma(s)\,dW_\nu(s), \qquad 0 \le t \le T,
\end{aligned}
\tag{3.4$'$}
$$

is a nonnegative, $\mathbb{P}^\nu$-local martingale. Reasoning as in Remark 3.3 we deduce that (3.9) and (3.10) with $c(\cdot) \equiv \hat{c}(\cdot)$ hold here as well, and that the values of the portfolio $\hat{\pi}(\cdot)$ on $[\![S,T]\!]$ are irrelevant if $X(\cdot)$ is now the process of (6.7) and S the stopping time of (3.8).

In particular, for $\nu \equiv 0$, (6.13) gives

$$
\begin{aligned}
d\big(\gamma_0(t)\hat{X}(t)\big) &= -\gamma_0(t)\,d\hat{c}(t) + \gamma_0(t)\hat{X}(t)\hat{\pi}^*(t)\sigma(t)\,dW_0(t),\\
&\qquad \hat{X}(0) = V(0), \qquad \hat{X}(T) = B,
\end{aligned}
\tag{6.13$'$}
$$

which is (3.1) for the process $\hat{X}(\cdot)$ of (6.7). This shows $\hat{X}(\cdot) \equiv X^{V(0),\hat{\pi},\hat{c}}(\cdot)$ and hence $h(0) \le V(0) < \infty$.

To complete the proof, it thus suffices to show $h(0) \ge V(0)$. Clearly, we may assume $h(0) < \infty$, and then there exists a number $x \in (0,\infty)$ such that $X^{x,\pi,c}(T) \ge B$ a.s. for some $(\pi, c) \in \mathscr{A}'(x)$. Then the analogue of (6.13) holds and it follows from the supermartingale property that

$$
\begin{aligned}
x &\ge E^\nu\bigg[\gamma_\nu(T)X^{x,\pi,c}(T) + \int_0^T \gamma_\nu(t)\,dc(t)\\
&\qquad + \int_0^T \gamma_\nu(t)X^{x,\pi,c}(t)\{\delta(\nu(t)) + \nu^*(t)\pi(t)\}\,dt\bigg]\\
&\ge E^\nu[B\gamma_\nu(T)],
\end{aligned}
\tag{6.14}
$$

$\forall\, \nu \in \mathscr{D}$. Therefore, $x \ge V(0)$ and thus $h(0) \ge V(0)$. □

6.5 Definition. We say that a K-hedgeable contingent claim B is K-*attainable* if there exists a portfolio process π with values in K such that $(\pi, 0) \in \mathscr{A}'(V(0))$ and $X^{V(0),\pi,0}(T) = B$ a.s.

6.6 THEOREM. *For a given K-hedgeable contingent claim B and any given* $\lambda \in \mathscr{D}$, *the conditions*

$$\left\{ Q_\lambda(t) = V(t) \exp\left(-\int_0^t \delta(\lambda(u))\, du \right), \mathscr{F}_t; 0 \le t \le T \right\} \tag{6.15}$$

is a $\mathbb{P}^\lambda$*-martingale,*

$$\lambda \textit{ achieves the supremum in } V(0) = \sup_{\nu \in \mathscr{D}} E^\nu[B\gamma_\nu(T)], \tag{6.16}$$

and

$$\left\{ \begin{array}{c} B \textit{ is K-attainable (by a portfolio } \pi) \textit{ and the corresponding} \\ \gamma_\lambda(\cdot) X^{V(0), \pi, 0}(\cdot) \textit{ is a } \mathbb{P}^\lambda\textit{-martingale} \end{array} \right\} \tag{6.17}$$

are equivalent, and imply

$$\hat{c}(t, \omega) = 0, \delta(\lambda(t, \omega)) + \lambda^*(t, \omega)\hat{\pi}(t, \omega) = 0, \qquad \ell \otimes P\text{-a.e.} \tag{6.18}$$

for the pair $(\hat{\pi}, \hat{c}) \in \mathscr{A}'(V(0))$ *of Theorem* 6.4.

PROOF. The $\mathbb{P}^\lambda$-supermartingale $Q_\lambda(\cdot)$ is a $\mathbb{P}^\lambda$-martingale if and only if $Q_\lambda(0) = E^\lambda Q_\lambda(T) \Leftrightarrow V(0) = E^\lambda[B\gamma_\lambda(T)] \Leftrightarrow$ (6.16).

On the other hand, (6.15) implies $A_\lambda(\cdot) \equiv 0$, and so from (6.10), $\hat{c}(t) = -\int_0^t X(s)[\delta(\lambda(s)) + \lambda^*(s)\hat{\pi}(s)]\, ds$. Now (6.18) follows from the increase of $\hat{c}(\cdot)$ and the nonnegativity of $\delta(\lambda) + \lambda^*\hat{\pi}$, because $\hat{\pi}$ takes values in K.

From (6.16) [and its consequences (6.15) and (6.18)], the process $\hat{X}(\cdot)$ of (6.7) and (6.13) coincides with $X^{V(0), \hat{\pi}, 0}(\cdot)$ and we have: $\hat{X}(T) = B$ almost surely, and $\gamma_\lambda(\cdot)\hat{X}(\cdot)$ is a $\mathbb{P}^\lambda$-martingale. Thus (6.17) is satisfied with $\pi \equiv \hat{\pi}$. On the other hand, suppose that (6.17) holds. Then $V(0) = E^\lambda[B\gamma_\lambda(T)]$, so (6.16) holds. □

6.7 THEOREM. *Let B be a K-hedgeable contingent claim. Suppose that for any* $\nu \in \mathscr{D}$ *with* $\delta(\nu) + \nu^*\hat{\pi} \equiv 0$,

$$Q_\nu(\cdot) \textit{ in } (6.4) \textit{ is of class } D[0, T], \textit{ under } \mathbb{P}^\nu. \tag{6.19}$$

Then, for any given $\lambda \in \mathscr{D}$, *the conditions* (6.15), (6.16) *and* (6.18) *are equivalent, and imply*

$$\left\{ \begin{array}{c} B \textit{ is K-attainable (by a portfolio } \pi) \textit{ and the corresponding} \\ \gamma_0(\cdot) X^{V(0), \pi, 0}(\cdot) \textit{ is a } \mathbb{P}^0\textit{-martingale} \end{array} \right\}. \tag{6.17'}$$

PROOF. We have already shown the implications (6.15) ⇔ (6.16) ⇔ (6.18). To prove that these three conditions are actually *equivalent* under (6.19), suppose that (6.18) holds. Then from (6.10), $A_\lambda(\cdot) \equiv 0$, whence the $\mathbb{P}^\lambda$-local martingale $Q_\lambda(\cdot)$ is actually a $\mathbb{P}^\lambda$-martingale [from (6.6) and the assumption (6.19)]. Thus (6.15) is satisfied.

Clearly then, if (6.15), (6.16) and (6.18) are satisfied for some $\lambda \in \mathscr{D}$, they are satisfied for $\lambda \equiv 0$ as well, and from Theorem 6.6 we know then that (6.17′) [i.e., (6.17) with $\lambda \equiv 0$] holds. □

6.8 REMARK. The conditions $V(0) < \infty$ and (6.19) are satisfied (the latter, in fact, for every $\nu \in \mathscr{D}$) in the case of the simple European call option $B = (P_1(T) - q)^+$, provided

$$\text{(6.20)} \qquad \text{the function } x \mapsto \delta(x) + x_1 \text{ is bounded from below on } \tilde{K}.$$

The same is true for any contingent claim B that satisfies $B \le \alpha P_1(T)$ a.s., for some $\alpha \in (0, \infty)$.

Indeed, with $\mu \in \mathscr{D}$ fixed, we have from (6.3) in this case, for any $\tau \in \mathscr{S}$,

$$V(\tau) \le \alpha \operatorname*{ess\,sup}_{\nu \in \mathscr{D}} E^{\nu}\left[P_1(T)\gamma_0(T) \exp\left(- \int_{\tau}^{T} \delta(\nu(u))\, du \right) \middle| \mathscr{F}_{\tau} \right] \quad \text{a.s.,}$$

where, without loss of generality, we may take the supremum over all $\nu \in \mathscr{D}$ that agree with μ on $[\![0, \tau]\!]$. Now

$$\text{(6.21)} \qquad \begin{aligned} &\exp\left(\int_0^t \nu_1(s)\, ds \right) \gamma_0(t) P_1(t) \\ &\quad = P_1(0) \exp\left\{ \int_0^t \sigma_1(s)\, dW_{\nu}(s) - \frac{1}{2} \int_0^t \| \sigma_1(s) \|^2\, ds \right\} \end{aligned}$$

is a $\mathbb{P}^{\nu}$-martingale, for every $\nu \in \mathscr{D}$. Thus

$$\begin{aligned} 0 &\le V(\tau) \exp\left(- \int_0^{\tau} \delta(\mu(s))\, ds \right) \\ &\le \alpha \operatorname*{ess\,sup}_{\nu \in \mathscr{D}} E^{\nu}\left[\gamma_0(T) P_1(T) \exp\left(- \int_0^{T} \delta(\nu(s))\, ds \right) \middle| \mathscr{F}_{\tau} \right] \\ &\le \text{const} \exp\left(\int_0^{\tau} \mu_1(s)\, ds \right) \gamma_0(\tau) P_1(\tau) \\ &= \text{const}\, E^{\mu}\left[\gamma_0(T) P_1(T) \exp\left(\int_0^{T} \mu_1(s)\, ds \right) \middle| \mathscr{F}_{\tau} \right]. \end{aligned}$$

Clearly from this, the family $\{V(\tau)\exp(-\int_0^{\tau} \delta(\mu(s))\, ds)\}_{\tau \in \mathscr{S}}$ *is uniformly integrable under* $\mathbb{P}^{\mu}$.

6.9 REMARK. Note that condition (6.20) is indeed satisfied, if the convex set

$$\text{(6.20}'\text{)} \qquad K \text{ contains both the origin and the point } (1, 0, \dots, 0)$$

(and thus also the line segment adjoining these points), for then $x_1 + \delta(x) \ge x_1 + \sup_{0 \le \alpha \le 1}(-\alpha x_1) = x_1^+ \ge 0$, $\forall\, x \in \tilde{K}$. This is the case in the Examples 5.2 (i)–(iv), (vi), and (vii) with $1 \le \beta_1 \le \infty$.

6.10 REMARK. *If the condition (6.20) is not satisfied, we have* $V(0) = \infty$ *for the European call option* $B = (P_1(T) - q)^+$ *with* $\delta(\cdot) \ge 0$, $r(\cdot) \ge 0$. In other words, such constraints make impossible the hedging of this contingent claim, starting with a finite initial capital.

Indeed, from Jensen's inequality we have $E^\nu[\gamma_\nu(T)(P_1(T) - q)^+] \geq (E^\nu[\gamma_\nu(T)P_1(T)] - q)^+$. But from (6.21),

$$E^\nu[\gamma_\nu(T)P_1(T)] = E^\nu\bigg[\exp\bigg(-\int_0^T(\delta(\nu(s)) + \nu_1(s))\,ds\bigg)$$
$$\times\exp\bigg\{\int_0^T \sigma_1(s)\,dW_\nu(s) - \frac{1}{2}\int_0^T\|\sigma_1(s)\|^2\,ds\bigg\}\bigg]$$

and for deterministic ν,

$$E^\nu[\gamma_\nu(T)P_1(T)] = \exp\bigg(-\int_0^T(\delta(\nu(s)) + \nu_1(s))\,ds\bigg).$$

Thus, with $\mathscr{D}_d$ denoting the class of nonrandom functions in $\mathscr{D}$, we have

$$\begin{aligned} V(0) &= \sup_{\nu\in\mathscr{D}} E_\nu\big[\gamma_\nu(T)(P_1(T) - q)^+\big] \geq \sup_{\nu\in\mathscr{D}}\big(E^\nu[\gamma_\nu(T)P_1(T)] - q\big)^+ \\ &\geq \sup_{\nu\in\mathscr{D}_d}\bigg(\exp\bigg\{-\int_0^T(\delta(\nu(s)) + \nu_1(s))\,ds\bigg\} - q\bigg)^+ = \infty. \end{aligned} \tag{6.22}$$

The conditions (6.20) and (6.20′) fail, for instance, in the case of *rectangular constraints* $K = \times_{i=1}^d[\alpha_i, \beta_i]$ of Example 4.2(vii) with $\beta_1 < 1$. They also fail in the case of an incomplete market in which investment in the first stock is prohibited, say with $K = \{\pi \in \mathscr{R}^d;\ \pi_1 = \cdots = \pi_m = 0\}$ for some $1 \leq m \leq d$. Then $\tilde{K} = \{x \in \mathscr{R}^d;\ x_{m+1} = \cdots = x_d = 0\}$ and $\delta \equiv 0$ on $\tilde{K}$. (However, the "option with a ceiling" $B = \min\{(P_1(T) - q)^+, L\}$ for some real $L > 0$ is bounded, thus hedgeable, for any constraint set K.)

In El Karoui and Quenez (1993) such problems are avoided by assuming a priori the claim B to be hedgeable. Moreover, these authors work under the (difficult to verify) condition, that the hedging of B can be done by a portfolio $\pi(\cdot)$ for which the process $\pi(\cdot)X(\cdot)$ is bounded.

6.11 Remark. A slight modification of the proof of Theorem 6.4 shows that

$$V(0) = \sup_{\nu\in\mathscr{D}} E^\nu[B\gamma_\nu(T)] = \sup_{\nu\in\mathscr{H}} E[B\gamma_\nu(T)Z_\nu(T)] \tag{6.23}$$

holds for an arbitrary contingent claim B. The straightforward details are left to the diligence of the reader.

6.12 Remark. As a referee points out, it is not necessary to think of the process $\hat{c}$ as cumulative consumption. Instead, it can represent, for example, the cumulative cash surplus generated by the hedging portfolio $\hat{\pi}$, and this surplus might be reinvested in the market rather than consumed.

The next result characterizes the process $\hat{X}(\cdot)$ of (6.7) as the *minimal* solution of a certain *backwards stochastic differential equation* (BSDE) *with convex constraints*. It would be of considerable independent interest to develop a general theory for such equations, perhaps by "combining" our approach with that of Pardoux and Peng (1990).

6.13 PROPOSITION. *Suppose we have* $V(0) < \infty$, *and let* (X, π, c) *be any triple of* $\mathscr{R} \times \mathscr{R}^d \times [0, \infty)$*-valued, adapted process, such that* $c(\cdot)$ *has increasing, RCLL paths,* $c(T) + \int_0^T \|\pi(s)\|^2\, ds < \infty$ *a.s., and such that the BSDE*

$$X(t) = B + (c(T) - c(t)) \tag{6.24}$$
$$- \int_t^T X(s)\left[r(s)\,ds + \pi^*(s)\sigma(s)\,dW_0(s)\right], \qquad 0 \le t \le T$$

and the convex constraint

$$(X(t), \pi(t)) \in [0,\infty) \times K, \qquad 0 \le t \le T \tag{6.25}$$

are satisfied almost surely. Then the triple $(\hat{X}, \hat{\pi}, \hat{c})$ *of the Theorem* 6.4 *solves the problem* (6.24), (6.25), *and we have* $\hat{X}(\cdot) \le X(\cdot)$, *a.s.*

PROOF. The first claim follows directly from (6.13)′. For an arbitrary solution (X, π, c) of (6.24), (6.25), Itô's rule applied to the analogue

$$M_\nu(t) \triangleq \gamma_\nu(t)X(t) + \int_0^t \gamma_\nu(s)\,dc(s)$$
$$+ \int_0^t \gamma_\nu(s)X(s)\left[\delta(\nu(s)) + \nu^*(s)\pi(s)\right]ds$$

of (3.4)′, exhibits this nonnegative process as a local $\mathbb{P}^\nu$-martingale $M_\nu(t) = X(0) + \int_0^t \gamma_\nu(s)X(s)\pi^*(s)\sigma(s)\,dW_\nu(s)$, hence a *nonnegative* $\mathbb{P}^\nu$*-supermartingale*, for any $\nu \in \mathscr{D}$. This last property is inherited by the process

$$\gamma_\nu(t)X(t) = \gamma_0(t)X(t)\exp\left\{-\int_0^t \delta(\nu(s))\,ds\right\}, \qquad 0 \le t \le T,$$

for all $\nu \in \mathscr{D}$. But from Proposition 6.3 and (6.7), we have then

$$\gamma_0(\cdot)X(\cdot) \ge V(\cdot) = \gamma_0(\cdot)\hat{X}(\cdot), \quad \text{a.s.} \qquad \square$$

7. Examples. Let us now illustrate the results of Section 6 by means of some simple examples.

7.1 EXAMPLE (No short-selling). In the case $d = 1$, $K = [0, \infty)$ of Example 5.2(ii) and with $r, \sigma \equiv \sigma_{11}$ positive constants, we have $\tilde{K} = K$, $\delta(x) = 0$ for $x \ge 0$, $\delta(x) = \infty$ for $x < 0$, and so $x + \delta(x) = x \ge 0$ on $\tilde{K}$. Now take $B = \varphi(P_1(T))$, where $\varphi\colon \mathscr{R}^+ \to [0, \infty)$ is continuous, increasing, piecewise continuously differentiable and satisfies $\varphi(p) \le \alpha p$ for some real $\alpha > 0$. Then we have from Remark 6.8 that $V(0) < \infty$ and that condition (6.19) is satisfied. In fact, from the "classical" theory of Section 4 (cf. Example 4.3), we know that

$$X(t) = e^{rt}V(t) = e^{-r(T-t)}U(T - t, P_1(t)), \tag{7.1}$$

$$\hat{\pi}(t) = \frac{Q(T - t, P_1(t))}{U(T - t, P_1(t))} = P_1(t)\frac{(\partial/\partial p)U(T - t, P_1(t))}{U(T - t, P_1(t))} \ge 0, \tag{7.2}$$

where

$$
(7.3)\quad
\begin{aligned}
U(t,p) &\triangleq \int_{-\infty}^{\infty} \varphi\left(pe^{\sigma(\xi+\delta t)}\right)\frac{e^{-\xi^2/2t}}{\sqrt{2\pi t}}\,d\xi,\\
Q(t,p) &\triangleq \int_{-\infty}^{\infty} \psi\left(pe^{\sigma(\xi+\delta t)}\right)\frac{e^{-\xi^2/2t}}{\sqrt{2\pi t}}\,d\xi = p\frac{\partial}{\partial p}U(t,p)
\end{aligned}
$$

with $\delta = (r/\sigma) - (\sigma/2)$ and $\psi(p) \triangleq p\varphi'(p) \geq 0$ [hence $Q(t,p) \geq 0$]. If $p\varphi'(p) \geq \varphi(p)$ holds everywhere on $\mathscr{R}_+$ and $p\varphi'(p) > \varphi(p)$ holds on a set of positive Lebesgue measure, then $p(\partial/\partial p)U(t,p) > U(t,p)$, whence

$$
(7.4)\qquad \hat{\pi}(t) > 1, \qquad 0 \leq t \leq T.
$$

For instance, this is the case of the European call option $\varphi(p) = (p-q)^+$ with exercise price $q > 0$. Thus, the unconstrained hedging portfolio does not require short-selling and the constraint $K = [0,\infty)$ makes no difference. In particular, the supremum $V(0)$ of (5.7) is achieved for $\nu \equiv 0$ and is equal to the unconstrained hedging price u_o of the option.

7.2 EXAMPLE (No borrowing). Let $d = 1$, $K = (-\infty, 1]$ as in Example 5.2(vi). Then $\tilde{K} = (-\infty, 0]$, $\delta(\nu) = -\nu$, and consider the contingent claim $B = (P_1(T) - q)^+$. From (6.21) of Remark 6.8 we know that the process $\exp(\int_0^t \nu(s)\,ds)\gamma_0(t)P_1(t)$ is a $\mathbb{P}^\nu$-martingale, for every $\nu \in \mathscr{D}$. Consequently,

$$
(7.5)\quad
\begin{aligned}
V(t) &\leq \operatorname*{ess\,sup}_{\nu\in\mathscr{D}} \exp\left(-\int_0^t \nu(s)\,ds\right)E^\nu\left[\exp\left(\int_0^T \nu(s)\,ds\right)\gamma_0(T)P_1(T)\middle|\mathscr{F}_t\right]\\
&= \gamma_0(t)P_1(t), \qquad 0 \leq t \leq T.
\end{aligned}
$$

On the other hand, in the notation of (6.22) we have by Jensen's inequality,

$$
(7.6)\quad
\begin{aligned}
V(t) &\geq \operatorname*{ess\,sup}_{\nu\in\mathscr{D}} \left\{\exp\left(-\int_0^t \nu(s)\,ds\right)E^\nu\left[\exp\left(\int_0^T \nu(s)\,ds\right)\gamma_0(T)P_1(T)\middle|\mathscr{F}_t\right]\right.\\
&\qquad\qquad \left.-E^\nu\left[\exp\left(\int_t^T \nu(s)\,ds\right)\gamma_0(T)q\middle|\mathscr{F}_t\right]\right\}^+\\
&\geq \operatorname*{ess\,sup}_{\nu\in\mathscr{D}_d} \left\{\gamma_0(t)P_1(t) - \exp\left(\int_t^T \nu(s)\,ds\right)qE^\nu[\gamma_0(T)|\mathscr{F}_t]\right\}^+\\
&= \gamma_0(t)P_1(t)
\end{aligned}
$$

for $0 \leq t < T$. The inequalities (7.5) and (7.6) imply

$$
(7.7)\qquad V(t) = \begin{cases} \gamma_0(t)P_1(t), & 0 \leq t < T,\\ \gamma_0(T)(P_1(T)-q)^+, & t = T, \end{cases}
$$

or equivalently

$$dV(t) = \gamma_0(t)P_1(t)\sigma(t)\,dW_0(t) - dA_0(t), \tag{7.8}$$

where

$$A_0(t) = \begin{cases} 0, & 0 \le t < T, \\ \gamma_0(T)\left[P_1(T) - (P_1(T) - q)^+\right], & t = T. \end{cases} \tag{7.9}$$

In particular, (7.8) implies $X \triangleq V/\gamma_0 \equiv X^{V(0),\hat{\pi},\hat{c}}$ with

$$\hat{\pi}(t) \equiv 1, \qquad \hat{c}(t) = \int_0^t \gamma_0^{-1}(s)\,dA_0(s). \tag{7.10}$$

In other words, in order to replicate $B = (P_1(T) - q)^+$ without borrowing, one has to invest all the wealth in the stock, not consume before the expiration date T, and consume at time $t = T$ the amount

$$\hat{c}(T) = P_1(T) - (P_1(T) - q)^+ = \min(P_1(T), q). \tag{7.11}$$

This example resolves two questions that can be raised in the context of Theorem 6.4. First, it shows that the process $V(\cdot)$ *is not, in general, a regular* $\mathbb{P}^0$*-supermartingale*, for if it were, $A_0(\cdot)$ would be continuous [e.g., Karatzas and Shreve (1988), page 28]. Second, it shows that, *in general, the supremum of* (5.7) *is not attained*. Indeed, one has to let $\nu \equiv -\infty$ in order to achieve equality in (7.6).

7.3 EXAMPLE (Option with a ceiling on a stock that cannot be traded). Let $K = \{x \in \mathscr{R}^d;\ x_1 = 0\}$, $B = (P_1(T) - q)^+ \wedge L$ for some real $q > 0$, $L > 0$. Then $\tilde{K} = \{x \in \mathscr{R}^d;\ x_2 = x_3 = \cdots = x_d = 0\}$ and $\delta \equiv 0$ on $\tilde{K}$. Assume deterministic market coefficients. We want to verify

$$V(0) = \gamma_0(T)L \tag{7.12}$$

by first showing $V(0) \ge \gamma_0(T)L$ and then proving the opposite inequality by constructing a consumption process c such that the wealth process corresponding to the triple $(\gamma_0(T)L, 0, c)$ satisfies $X(T) = B$ a.e. In the notation of (6.22), we have

$$V(0) \ge \gamma_0(T)L \operatorname*{ess\,sup}_{\nu \in \mathscr{D}_d} E^\nu 1_{\{P_1(T) - q > L\}}. \tag{7.13}$$

Define an $\mathscr{R}_+^d$-valued process $\tilde{P}^{(\nu)}(\cdot) = \{\tilde{P}_i^\nu(\cdot)\}_{i=1}^d$ by

$$\begin{aligned} d\tilde{P}_i^{(\nu)}(t) &= \tilde{P}_i^{(\nu)}(t)\left[r(t) - \nu_i(t)\right]dt \\ &\quad + \tilde{P}_i^{(\nu)}(t)\sum_{j=1}^d \sigma_{ij}(t)\,dW_0^{(j)}(t), \quad \tilde{P}_i^{(\nu)}(0) = P_i(0), \end{aligned} \tag{7.14}$$

for $i = 1, \ldots, d$ and $\nu \in \mathscr{D}_d$. Then a comparison of (7.14) with (5.14) shows that $\tilde{P}^{(\nu)}(\cdot)$ has the same distribution under $\mathbb{P}^0$ as $P(\cdot) = \{P_i(\cdot)\}_{i=1}^d$ has under

$\mathbb{P}^\nu$ and thus

$$E^\nu 1_{\{P_1(T)-q>L\}} = E^0 1_{\{\tilde{P}_1^{(\nu)}(T)-q>L\}}. \tag{7.15}$$

Letting $\nu \to -\infty$, (7.13)–(7.15) imply

$$V(0) \geq \gamma_0(T)L. \tag{7.16}$$

Next, define a consumption process c by

$$c(t) = \begin{cases} 0, & t < T \text{ or } t = T,\ P_1(T) - q > L, \\ L - (P_1(T) - q)^+, & t = T,\ P_1(T) - q \leq L. \end{cases} \tag{7.17}$$

Then the wealth process $X(\cdot)$ associated with the policy $(\gamma_0(T)L, 0, c)$ is given by $X(t) = (\gamma_0(T)/\gamma_0(t))L$ for $t < T$ and by

$$X(T) = L - c(T) = B \tag{7.18}$$

for $t = T$. This implies $V(0) \leq \gamma_0(T)L$ by Theorem 6.4 and, in conjunction with (7.16), leads to (7.12).

Consequently, the way to hedge a bounded option on a stock that is not available for investment is to replicate the upper bound of the option by investing in the bond only, and then to consume the difference at the expiration date.

8. Extensions and ramifications.

1. As in Section 16 of CK, we can let the constraint set K depend on $(t, \omega) \in [0, T] \times \Omega$ in a nonanticipative way.
2. The hedging price $V(0)$ can be regarded as an upper bound for the "fair price" of the contingent claim; in the terminology of El Karoui and Quenez (1993) it can be called the *selling price*. As in that paper, we could also consider a lower bound, or the *purchase price*, by replacing the *sup* operator by the *inf* operator.
3. As for numerical calculations, we refer again to El Karoui and Quenez (1993). It is shown in that paper, in the special case of "incompleteness" constraints and constant r, σ, that $V(0) = Q(0, p)$, where $Q(t, p)$ is the pointwise limit $Q(t, p) = \lim_{n\to\infty} Q^n(t, p)$. Here we consider a contingent claim of the form $B = \varphi(P(T))$, for an appropriate continuous function φ: $\mathscr{R}_+^d \to \mathscr{R}_+$ of the vector $P(t) \triangleq (P_1(t), \dots, P_d(t))^*$ of stock prices at the terminal time $t = T$, and define

$$Q^n(t, p) \triangleq \sup_{\nu \in \mathscr{D}_n} E^\nu\left[\varphi(P(T))\frac{\gamma_\nu(T)}{\gamma_\nu(t)}\middle| P(t) = p\right], 0 \leq t \leq T,\ p \in \mathscr{R}_+^d, \tag{8.1}$$

with $\mathscr{D}_n \triangleq \{\nu \in \mathscr{D};\ \|\nu(t, \omega\| \leq n,$ for $\ell \otimes \mathbb{P}$-a.e. $(t, \omega)\}$, $n \in \mathbb{N}$. Moreover, from (8.1) and the dynamics (5.14) of the process $P(\cdot)$ under $\mathbb{P}^\nu$, the value function Q^n of (8.1) can be characterized in terms of the following Cauchy problem for the associated HJB equation [cf. Fleming and Rishel (1975)]:

$$\begin{aligned}(8.2)\qquad &\frac{\partial Q^n}{\partial t} + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij} p_i p_j \frac{\partial^2 Q^n}{\partial p_i\,\partial p_j} + r\left(\sum_{i=1}^{d} p_i \frac{\partial Q^n}{\partial p_i} - Q^n\right)\\ &\quad + \max_{\nu\in\tilde K;\,\|\nu\|\le n}\left(-\sum_{i=1}^{d}\nu_i p_i \frac{\partial Q^n}{\partial p_i} - \delta(\nu)Q^n\right) = 0,\\ &0 \le t < T,\ p\in\mathscr{R}_+^d,\quad Q^n(T,p) = \varphi(p),\quad t = T,\ p\in\mathscr{R}_+^d.\end{aligned}$$

9. Hedging claims with higher interest rate for borrowing. We have studied so far a model in which one is allowed to borrow money at an interest rate $R(\cdot)$ equal to the bond rate $r(\cdot)$. In this section we consider the more general case of a financial market $\mathscr{M}^*$ in which $R(\cdot) \ge r(\cdot)$ without constraints on portfolio choice. We assume that the progressively measurable process $R(\cdot)$ is also bounded.

In this market $\mathscr{M}^*$ it is not reasonable to borrow money and to invest money in the bond at the same time. Therefore, we restrict ourselves to policies for which the relative amount borrowed at time t is equal to $(1-\sum_{i=1}^d \pi_i(t))^-$. Then, as in Section 18 of CK, the wealth process $X = X^{x,\pi,c}$ corresponding to initial capital $x > 0$ and portfolio–cumulative consumption pair (π, c) as in Definition 3.1, satisfies

$$\begin{aligned}dX(t) &= r(t)X(t)\,dt - dc(t)\\ (9.1)\qquad &\quad + X(t)\left[\pi^*(t)\sigma(t)\,dW_0(t) - (R(t)-r(t))\left(1-\sum_{i=1}^{d}\pi_i(t)\right)^{-} dt\right].\end{aligned}$$

We set $\delta(\nu(t)) = -\nu_1(t)$ for $\nu\in\mathscr{D}$, where

$$(9.2)\qquad \begin{aligned}\mathscr{D} \triangleq \{&\nu;\ \nu \text{ progressively measurable, } \mathscr{R}^d\text{-valued process with}\\ &r - R \le \nu_1 = \cdots = \nu_d \le 0,\ \ell\otimes\mathbb{P}\text{-a.e.}\}.\end{aligned}$$

With this notation, *the theory of the previous sections goes then through with only minor changes*. For instance, in Theorem 6.6 we have to replace (6.18) by

$$(9.3)\qquad \hat c(t,\omega) \equiv 0,\qquad \Psi^{\lambda,\hat\pi}(t,\omega) = 0,\qquad \ell\otimes\mathbb{P}\text{-a.e.},$$

where $(\hat\pi, \hat c)$ is the portfolio–consumption process pair of Theorem 6.4 and

$$\begin{aligned}(9.4)\qquad \Psi^{\nu,\pi}(t) &\triangleq [R(t) - r(t) + \nu_1(t)]\left(1-\sum_{i=1}^{d}\pi_i(t)\right)^{-}\\ &\quad - \nu_1(t)\left(1-\sum_{i=1}^{d}\pi_i(t)\right)^{+},\quad 0\le t\le T,\end{aligned}$$

is a nonnegative process. Similarly, Theorem 6.7 now takes the following form.

6.7* THEOREM. *Let $\hat{\pi}$ be the portfolio process of Theorem* 6.4, *and suppose that* (6.19) *holds for every $\nu \in \mathscr{D}$ that satisfies $\Psi^{\nu,\hat{\pi}} \equiv 0$. Then, for any given $\lambda \in \mathscr{D}$, the conditions* (6.15), (6.16) *and* (9.3) *are equivalent and imply*

$$\left\{\begin{array}{l} B \text{ is attainable } (\text{by a portfolio } \pi) \\ \text{and the corresponding process} \\ \gamma_{\hat{\lambda}} X^{V(0),\pi,0}(\cdot) \text{ is a } \mathbb{P}^{\hat{\lambda}}\text{-martingale} \end{array}\right\} \tag{9.5}$$

for the process $\hat{\lambda} \in \mathscr{D}$ given by (9.6). *Conversely, if* (9.5) *holds, then the conditions* (6.15), (6.16) *and* (9.3) *are satisfied for some $\lambda \in \mathscr{D}$; in particular, for*

$$\hat{\lambda}(t) = \hat{\lambda}_1(t)\mathbf{1}, \qquad \hat{\lambda}_1(t) \triangleq [r(t) - R(t)]1_{\{\sum_{i=1}^d \hat{\pi}_i(t) > 1\}}. \tag{9.6}$$

Actually, in this case, we have the following *existence result* under a condition analogous to (6.19).

9.1 THEOREM. *If the process $Q_{\hat{\lambda}}(\cdot)$ of* (6.15) *is of class $D[0,T]$ under $\mathbb{P}^{\hat{\lambda}}$, with $\hat{\lambda}$ as in* (9.6), *then $\hat{\lambda}$ is optimal; namely,* (6.16) *holds for $\lambda = \hat{\lambda}$.*

To prove Theorem 9.1, we need the following lemmas.

9.2 LEMMA. *The set $\mathscr{Z} \triangleq \{Z_\nu(\cdot);\ \nu \in \mathscr{D}\}$ is a convex set of real-valued processes defined on $[0,T]$.*

9.3 LEMMA. *The set $\mathscr{Z}_T \triangleq \{Z_\nu(T);\ \nu \in \mathscr{D}\}$ is bounded in $L^2(\mathbb{P})$.*

9.4 LEMMA. *$\mathscr{Z}_T$ is strongly closed in $L^2(\mathbb{P})$.*

For Lemma 9.2 see, for example, Lemma 2.2 in Xu (1990). Lemma 9.3 follows from

$$Z_\nu^2(T) = \exp\left\{-\int_0^T 2\theta_\nu(t)\,dW(t) - \frac{1}{2}\int_0^T \|2\theta_\nu(t)\|^2\,dt + \int_0^T \|\theta_\nu(t)\|^2\,dt\right\}$$

and the boundedness of $\theta_\nu(\cdot)$. Lemma 9.4 is a consequence of Theorem 4 in Beneš (1971).

PROOF OF THEOREM 9.1. Let $\{\nu_n;\ n \in \mathbb{N}\}$ be a maximizing sequence in (5.7); that is, $\lim_{n\to\infty} E^{\nu_n}[\gamma_{\nu_n}(T)B] = V(0)$. By analogy with (6.14), we get

$$E^{\nu_n}\left[\gamma_{\nu_n}(T)B + \int_0^T \gamma_{\nu_n}(t)\,d\hat{c}(t) + \int_0^T \gamma_{\nu_n}(t)\hat{X}(t)\Psi^{\nu_n,\hat{\pi}}(t)\,dt\right] \le V(0), \qquad \forall\, n \in \mathbb{N},$$

with $\Psi^{\nu_n,\hat{\pi}}$ as in (9.4) and $\hat{c}, \hat{\pi}, \hat{X}$ as in Theorem 6.4. This implies $\lim_{n\to\infty} E^{\nu_n}\int_0^T \gamma_{\nu_n}(t)\,d\hat{c}(t) = 0$ and, because the family of processes $\{\gamma_{\nu_n}(\cdot);\ n \in \mathbb{N}\}$ is bounded away from zero, $\lim_{n\to\infty} E[Z_{\nu_n}(T)\hat{c}(T)] = 0$. By Lemmas 9.2–9.4, the set $\mathscr{Z}_T$ is weakly compact in $L^2(\mathbb{P})$; hence, there exists $\nu \in \mathscr{D}$ and a (relabelled) subsequence $\{\nu_n;\ n \in \mathbb{N}\}$, such that $\lim_{n\to\infty} E[Z_{\nu_n}(T)\hat{c}(T)] = E[Z_\nu(T)\hat{c}(T)] = 0$. It follows that $\hat{c}(t,\omega) = 0$, $\ell \otimes \mathbb{P}$-a.e. and thus $\hat{\lambda}$ of (9.6) is optimal by Theorem 6.7*. □

Theorem 9.1 implies that, under its conditions, the contingent claim B is attainable in the market $\mathscr{M}^*$ with different interest rates for borrowing and lending. In the case $d = 1$, $B = \varphi(P_1(T))$ with $\varphi\colon \mathscr{R}_+ \to [0,\infty)$ as in Example 7.1 and with constant $R > r$, the condition of Theorem 9.1 is actually satisfied (cf. Remark 6.8). If $p\varphi'(p) \ge \varphi(p)$ holds everywhere on $\mathscr{R}_+$ and strictly on a set of positive measure, then we may take $\hat{\lambda} \equiv r - R$ and the Black–Scholes formulae (7.1) and (7.3) remain valid if we replace in them r by R. This follows from (7.4), which can be shown, in the present context, either directly, or as in the following example.

9.5 Example. Let us consider the case of constant coefficients $r, R, \{\sigma_{ij}\} = \sigma$. Then the vector $P(t) = (P_1(t), \dots, P_d(t))$ of stock price processes satisfies the equations

$$
\begin{aligned}
dP_i(t) &= P_i(t)\left[b_i(t)\,dt + \sum_{i=1}^{d} \sigma_{ij}\,dW^{(j)}(t)\right] \\
&= P_i(t)\left[(r - \nu_1(t))\,dt + \sum_{i=1}^{d} \sigma_{ij}\,dW_\nu^{(j)}(t)\right], \qquad 1 \le i \le d,
\end{aligned} \tag{9.7}
$$

for every $\nu \in \mathscr{D}$ [recall (2.2) and (5.14)]. Consider now a contingent claim of the form $B = \varphi(P(T))$ for a given continuous function $\varphi\colon \mathscr{R}_+^d \to [0,\infty)$ that satisfies a polynomial growth condition, as well as the *value function*

$$
Q(t,p) \triangleq \sup_{\nu \in \mathscr{D}} E^\nu\left[\varphi(P(T))\exp\left(-\int_t^T (r - \nu_1(s))\,ds\right)\middle| P(t) = p\right] \tag{9.8}
$$

on $[0,T] \times \mathscr{R}_+^d$. Clearly, the processes X and V of (6.7) and (6.2) are given as

$$
\hat{X}(t) = Q(t, P(t)), \qquad V(t) = e^{-rt}\hat{X}(t), \qquad 0 \le t \le T,
$$

where Q solves the semilinear parabolic partial differential equation of the Hamilton–Jacobi–Bellman (HJB) type,

$$
\begin{aligned}
&\frac{\partial Q}{\partial t} + \frac{1}{2}\sum_i \sum_j a_{ij} p_i p_j \frac{\partial^2 Q}{\partial p_i\,\partial p_j} \\
&\quad + \max_{r - R \le \nu_1 \le 0}\left[(r - \nu_1)\left\{\sum_i p_i \frac{\partial Q}{\partial p_i} - Q\right\}\right] = 0, \quad 0 \le t < T,\ p \in \mathscr{R}_+^d, \\
&Q(T,p) = \varphi(p), \qquad p \in \mathscr{R}_+^d,
\end{aligned} \tag{9.9}
$$

associated with the control problem of (9.8) and the dynamics (9.7) [cf. Ladyženskaja, Solonnikov and Ural'tseva (1968) for the basic theory of such equations, and Fleming and Rishel (1975) for the connections with stochastic control]. Clearly, the maximization in (9.9) is achieved by $\nu_1^* = -(R-r)1_{\{\Sigma_i p_i(\partial Q/\partial p_i) \geq Q\}}$. The portfolio $\hat{\pi}(\cdot)$ of Theorem 6.7* and the process $\hat{\lambda}_1(\cdot)$ of (9.6) are then given, respectively, by

$$\hat{\pi}_i(t) = \frac{P_i(t)(\partial/\partial p_i)Q(t,P(t))}{Q(t,P(t))}, \qquad i = 1,\dots,d, \tag{9.10}$$

and

$$\hat{\lambda}_1(t) = (r-R)1_{\{\Sigma_i \hat{\pi}_i(t) \geq 1\}}. \tag{9.11}$$

Suppose now that the function φ satisfies $\Sigma_i\, p_i(\partial\varphi(p)/\partial p_i) \geq \varphi(p)$, $\forall\, p \in \mathscr{R}_+^d$. Then the solution Q of (9.9) also satisfies the inequality

$$\sum_i p_i \frac{\partial Q(t,p)}{\partial p_i} \geq Q(t,p), \qquad 0 \leq t \leq T, \tag{9.12}$$

for all $p \in \mathscr{R}_+^d$, and is actually given explicitly as

$$Q(t,p) = E^{(r-R)1}\left[e^{-R(T-t)}\varphi(P(T)) \mid P(t) = p\right]$$

$$= \begin{cases} e^{-R(T-t)}\displaystyle\int_{R^d}\varphi(h(T-t,p,\sigma z;R))(2\pi t)^{-d/2}\exp\left(-\frac{\|z\|^2}{2t}\right)dz, \\ \hfill 0 \leq t < T,\ p \in \mathscr{R}_+^d, \\ \varphi(p), \qquad t = T,\ p \in R_+^d, \end{cases} \tag{9.13}$$

in the notation of (4.7) (recall Example 4.3). Indeed, it is straightforward to check that, in this case, the function of (9.13) satisfies the inequality (9.12), as well as the linear PDE

$$\frac{\partial Q}{\partial t} + \frac{1}{2}\sum_i\sum_j p_i p_j a_{ij}\frac{\partial^2 Q}{\partial p_i\,\partial p_j} + R\left(\sum_i p_i \frac{\partial Q}{\partial p_i} - Q\right) = 0, \quad 0 \leq t < T,\ p \in \mathscr{R}_+^d,$$

$$Q(T,p) = \varphi(p), \qquad p \in \mathscr{R}_+^d,$$

and thus also the nonlinear equation (9.9). In this case the portfolio $\hat{\pi}(\cdot)$ always borrows: $\Sigma_{i=1}^d \hat{\pi}_i(t) \geq 1$, $0 \leq t \leq T$ (a.s.), and thus $\hat{\lambda}_1(t) = r - R$, $0 \leq t \leq T$.

9.6 Remark. The pair of *adapted* processes $(\hat{X}, \hat{\pi})$, with $\hat{X}(\cdot) \equiv X^{x,\hat{\pi},0}(\cdot)$ as in Theorem 6.7*, satisfies

$$d\hat{X}(t) = \hat{X}(t)\left[r(t) - (R(t) - r(t))\left(1 - \sum_{i=1}^d \hat{\pi}_i(t)\right)^-\right]dt$$

$$+\hat{X}(t)\hat{\pi}^*(t)\sigma(t)\,dW_0(t), \qquad 0 \leq t \leq T,$$

$$\hat{X}(T) = B,$$

almost surely. This is a *nonlinear backwards stochastic differential equation* in the spirit of Pardoux and Peng (1990); it admits an "explicit" solution in the context of Example 9.5.

9.7 REMARK. We can also study the combined problem of hedging under constraints *and* with higher interest rate for borrowing than for lending. In that case, the hedging price can be shown to be equal to $\sup_{(\nu,\mu)\in\mathscr{D}_1\times\mathscr{D}_2} E[H_{\nu,\mu}(T)B]$, where $\mathscr{D}_1$ is the set $\mathscr{D}$ of Section 5, $\mathscr{D}_2$ is the set of (9.2) in this section, $\theta_{\nu,\mu}(t) \triangleq \theta(t) + \sigma^{-1}(t)[\nu(t) + \mu(t)]$ and

$$H_{\nu,\mu}(t) \triangleq \exp\biggl[-\int_0^t \{r(s) + \delta(\nu(s)) - \mu_1(s)\}\,ds - \int_0^t \theta^*_{\nu,\mu}(s)\,dW(s) - \frac{1}{2}\int_0^t \|\theta_{\nu,\mu}(s)\|^2\,ds\biggr].$$

APPENDIX

We present in this Appendix the technical proofs of Propositions 6.2 and 6.3.

PROOF OF PROPOSITION 6.2. Let us start by observing that, for any $\theta \in \mathscr{S}$, the random variable

$$\begin{aligned} J_\nu(\theta) &\triangleq E^\nu\biggl[V(T)\exp\biggl(-\int_\theta^T \delta(\nu(s))\,ds\biggr)\bigg|\mathscr{F}_\theta\biggr] \\ &= \frac{E\bigl[Z_\nu(\theta)Z_\nu(\theta,T)V(T)\exp\bigl(-\int_\theta^T \delta(\nu(s))\,ds\bigr)|\mathscr{F}_\theta\bigr]}{E\bigl[Z_\nu(\theta)Z_\nu(\theta,T)|\mathscr{F}_\theta\bigr]} \\ &= E\biggl[Z_\nu(\theta,T)V(T)\exp\biggl(-\int_\theta^T \delta(\nu(s))\,ds\biggr)\bigg|\mathscr{F}_\theta\biggr] \end{aligned}$$

depends only on the restriction of ν to $[\![\theta, T]\!]$ [we have used the notation $Z_\nu(\theta,T) = (Z_\nu(T)/Z_\nu(\theta))$]. It is also easy to check that the family of random variables $\{J_\nu(\theta)\}_{\nu\in\mathscr{D}}$ is directed upward; indeed, for any $\mu \in D$, $\nu \in \mathscr{D}$ and with $A = \{(t, w); \mathscr{T}_\mu(t, w) \geq \mathscr{T}_\nu(t, w)\}$ the process $\lambda \triangleq \mu 1_A + \nu 1_{A^c}$ belongs to $\mathscr{D}$ and we have a.s. $\mathscr{T}_\lambda(\theta) = \min\{\mathscr{T}_\mu(\theta), \mathscr{T}_\nu(\theta)\}$; then from Neveu (1975), page 121, there exists a sequence $\{\nu_k\}_{k\in\mathbb{N}} \subseteq \mathscr{D}$ such that $\{J_{\nu_k}(\theta)\}_{k\in\mathbb{N}}$ is increasing and

$$V(\theta) = \lim_{k\to\infty} \uparrow J_{\nu_k}(\theta) \quad \text{a.s.} \tag{A.1}$$

Returning to the proof itself, let us observe that

$$
\begin{aligned}
V(\tau) &= \operatorname*{ess\,sup}_{\nu \in \mathscr{D}_{\tau,T}} E^{\nu}\Bigg[\exp\bigg(-\int_{\tau}^{\theta}\delta(\nu(s))\,ds\bigg) \\
&\qquad\qquad \times E^{\nu}\bigg\{V(T)\exp\bigg(-\int_{\theta}^{T}\delta(\nu(s))\,ds\bigg)\bigg|\mathscr{F}_{\theta}\bigg\}\bigg|\mathscr{F}_{\tau}\Bigg] \\
&\le \operatorname*{ess\,sup}_{\nu \in \mathscr{D}_{\tau,T}} E^{\nu}\bigg[\exp\bigg(-\int_{\tau}^{\theta}\delta(\nu(s))\,ds\bigg)V(\theta)\bigg|\mathscr{F}_{\tau}\bigg] \quad \text{a.s.}
\end{aligned}
$$

To establish the opposite inequality, it certainly suffices to pick $\mu \in \mathscr{D}$ and show that

$$
V(\tau) \ge E^{\mu}\bigg[V(\theta)\exp\bigg(-\int_{\tau}^{\theta}\delta(\mu(s))\,ds\bigg)\bigg|\mathscr{F}_{\tau}\bigg] \tag{A.2}
$$

holds almost surely.

Let us denote by $M_{\tau,\theta}$ the class of processes $\nu \in \mathscr{D}$ that agree with μ on $[\![\tau, \theta]\!]$. We have

$$
\begin{aligned}
V(\tau) &\ge \operatorname*{ess\,sup}_{\nu \in M_{\tau,\theta}} E^{\nu}\bigg[\exp\bigg(-\int_{\tau}^{\theta}\delta(\nu(s))\,ds - \int_{\theta}^{T}\delta(\nu(s))\,ds\bigg)V(T)\bigg|\mathscr{F}_{\tau}\bigg] \\
&= \operatorname*{ess\,sup}_{\nu \in M_{\tau,\theta}} E^{\nu}\Bigg[\exp\bigg(-\int_{\tau}^{\theta}\delta(\nu(s))\,ds\bigg) \\
&\qquad\qquad \times E^{\nu}\bigg\{\exp\bigg(-\int_{\theta}^{T}\delta(\nu(s))\,ds\bigg)V(T)|\mathscr{F}_{\theta}\bigg\}\bigg|\mathscr{F}_{\tau}\Bigg].
\end{aligned}
$$

Thus, for every $\nu \in M_{\tau,\theta}$, we have

$$
\begin{aligned}
V(\tau) &\ge E^{\nu}\bigg[\exp\bigg(-\int_{\tau}^{\theta}\delta(\nu(s))\,ds\bigg)J_{\nu}(\theta)\bigg|\mathscr{F}_{\tau}\bigg] \\
&= \frac{E[Z_{\nu}(\tau)Z_{\nu}(\tau,\theta)E\{Z_{\nu}(\theta,T)|\mathscr{F}_{\theta}\}\exp(-\int_{\tau}^{\theta}\delta(\nu(s))\,ds)J_{\nu}(\theta)|\mathscr{F}_{\tau}]}{E[Z_{\nu}(\tau)Z_{\nu}(\tau,\theta)E\{Z_{\nu}(\theta,T)|\mathscr{F}_{\theta}\}|\mathscr{F}_{\tau}]} \\
&= E\bigg[Z_{\nu}(\tau,\theta)\exp\bigg(-\int_{\tau}^{\theta}\delta(\nu(s))\,ds\bigg)J_{\nu}(\theta)\bigg|\mathscr{F}_{\tau}\bigg] \\
&= E\bigg[Z_{\mu}(\tau,\theta)\exp\bigg(-\int_{\tau}^{\theta}\delta(\mu(s))\,ds\bigg)J_{\nu}(\theta)\bigg|\mathscr{F}_{\tau}\bigg] \\
&= \cdots = E^{\mu}\bigg[\exp\bigg(-\int_{\tau}^{\theta}\delta(\mu(s))\,ds\bigg)J_{\nu}(\theta)\bigg|\mathscr{F}_{\tau}\bigg].
\end{aligned}
$$

Now clearly we may take $\{\nu_k\}_{k\in\mathbb{N}} \subseteq M_{\tau,\theta}$ in (A.1), because $J_\nu(\theta)$ depends only on the restriction of ν on $[\![\theta, T]\!]$, and from the foregoing inequality,

$$\begin{aligned} V(\tau) &\geq \lim_{k\to\infty} \uparrow E^{\mu}\left[\exp\left(-\int_\tau^\theta \delta(\mu(s))\,ds\right) J_{\nu_k}(\theta)\middle|\mathscr{F}_\tau\right] \\ &= E^{\mu}\left[\exp\left(-\int_\tau^\theta \delta(\mu(s))\,ds\right)\lim_{k\to\infty}\uparrow J_{\nu_k}(\theta)\middle|\mathscr{F}_\tau\right] \\ &= E^{\mu}\left[\exp\left(-\int_\tau^\theta \delta(\mu(s))\,ds\right)V(\theta)\middle|\mathscr{F}_\tau\right] \quad \text{a.s.} \end{aligned}$$

by monotone convergence. □

It is an immediate consequence of this proposition that

$$\text{(A.3)}\qquad \begin{aligned} &V(\tau)\exp\left(-\int_0^\tau \delta(\nu(u))\,du\right) \\ &\quad\geq E^{\nu}\left[V(\theta)\exp\left(-\int_0^\theta \delta(\nu(u))\,du\right)\middle|\mathscr{F}_\tau\right] \quad \text{a.s.} \end{aligned}$$

holds for any given $\tau \in \mathscr{S}$, $\theta \in \mathscr{S}_{\tau,T}$ and $\nu \in \mathscr{D}$.

Proof of Proposition 6.3 [Adapted from El Karoui and Quenez (1993)]. Let us consider the positive, adapted process $\{V(t,\omega), \mathscr{F}_t;\ t \in [0,T] \cap \mathscr{Q}\}$ for $\omega \in \Omega$. From (A.3), the process

$$\left\{V(t,\omega)\exp\left(-\int_0^t \delta(\nu(s,\omega))\,ds\right), \mathscr{F}_t;\ t\in[0,T]\cap\mathscr{Q}\right\} \quad \text{for } \omega\in\Omega$$

is a $\mathbb{P}^\nu$-supermartingale on $[0,T]\cap\mathscr{Q}$, where $\mathscr{Q}$ is the set of rational numbers and thus has a.s. finite limits from the right and from the left [recall Proposition 1.3.14 in Karatzas and Shreve (1988), as well as the right-continuity of the filtration $\{\mathscr{F}_t\}$]. Therefore,

$$V(t+,\omega) \triangleq \begin{cases} \lim\limits_{\substack{s\downarrow t\\ s\in\mathscr{Q}}} V(s,\omega), & 0\leq t<T, \\ V(T,\omega), & t=T, \end{cases}$$

$$V(t-,\omega) \triangleq \begin{cases} \lim\limits_{\substack{s\uparrow t\\ s\in\mathscr{Q}}} V(s,\omega), & 0< t\leq T, \\ V(0), & t=0, \end{cases}$$

are well-defined and finite for every $\omega\in\Omega^*$, $\mathbb{P}(\Omega^*)=1$, and the resulting processes are adapted. Furthermore [Karatzas and Shreve (1988)], $\{V(t+)\exp(-\int_0^t \delta(\nu(s))\,ds), \mathscr{F}_t;\ 0\leq t\leq T\}$ is a RCLL, $\mathbb{P}^\nu$-supermartingale for all $\nu\in\mathscr{D}$. In particular,

$$V(t+) \geq E^{\nu}\left[V(T)\exp\left(-\int_t^T \delta(\nu(s))\,ds\right)\middle|\mathscr{F}_t\right] \quad \text{a.s.}$$

holds for every $\nu \in \mathscr{D}$, whence $V(t+) \geq V(t)$ a.s. On the other hand, from Fatou's lemma we have for any $\nu \in \mathscr{D}$,

$$\begin{aligned} V(t+) &= E^{\nu}\left[\lim_{n\to\infty} V\left(t+\frac{1}{n}\right)\exp\left(-\int_t^{t+1/n}\delta(\nu(u))\,du\right)\middle|\mathscr{F}_t\right] \\ &\leq \lim_{n\to\infty} E^{\nu}\left[V\left(t+\frac{1}{n}\right)\exp\left(-\int_t^{t+1/n}\delta(\nu(u))\,du\right)\middle|\mathscr{F}_t\right] \leq V(t) \quad \text{a.s.} \end{aligned}$$

and thus $\{V(t+),\mathscr{F}_t;\ 0 \leq t \leq T\}$ and $\{V(t),\mathscr{F}_t;\ 0 \leq t \leq T\}$ are modifications of one another.

The remaining claims are immediate. □

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